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A Fractional Fins Equation

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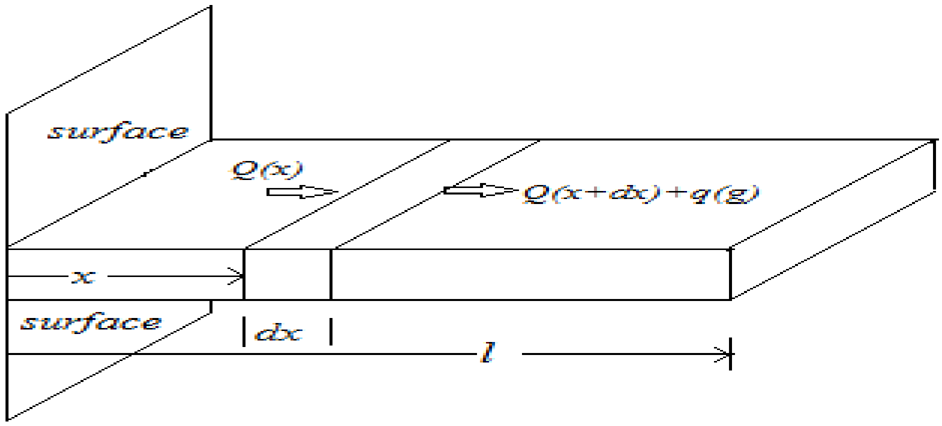
Abstract: -- Fractional order calculus can represent systems with higher- order dynamics and complex non linier phenomena using few coefficients, since the arbitrary order of the derivative provides an additional degree of freedom to fit a specific behavior. And other important characteristic is that fractional order derivatives depend not only on local conditions of the evaluated time but also on the entire history of the function. This fact is often useful when the system has a long- term "memory" and any evaluation point depends on the fast values of the function. The aim of present paper is finding the solution of fractional differential fin equation. Fins are frequently used in many heat transfer equations [1, 5, 10-12, 19, 21-22, 28]. The special case of fractional differential fin equation is same as the second order differential fin equation [29]. One of the objectives of this paper is to discuss the usefulness of fractional calculus in applied sciences and engineering.

Keywords and Phrases: Riemann-Liouville Fractional calculus operators, Solid fin, Heat transfer Natural convection, Mittag-Leffler function, Laplace transform.

Mathematics Subject Classification: 44A10, 44A15, 33C60, 82C31, 33E12.

1. INTRODUCTION:

Many engineering problems required high rate of heat transfer with reduced size and some engineering applications required lighter fin with higher rate of heat transfer where they use high thermal conductivity metals in applications such as airplane and motorcycle applications. The rate of heat transfer depends mainly on three parameters: Heat transfer coefficient (h), Availability of Surface area and Temperature difference between surface and surrounding fluid. The value of 'h' dependent on the properties of surrounding fluid and average velocity of fluid over the surface. Hence, it can be assumed as a constant in certain cases. Most of the times, the temperature difference is prescribed in a given application.



2. Governing Differential Equations:

Let us consider a straight rectangular fin protruding a wall surface in the figure. The characteristic dimensions of the fin are its length l, constant cross-sectional area A and circumference parameter ρ. Hence the heat balance equation for rectangular fin

$$Q(x) = q(c) + Q(x + dx) \tag{1}$$

We know the Fourier's law of conduction (heat conducted into the element at plane x)

$$Q(x) = -kA\frac{dt}{dx} \tag{2}$$

Heat conducted out of the element at plane $(x + dx)$

$$Q(x + dx) = Q(x) + \frac{dQ(x)}{dx}dx$$
$$= Q(x) - kA\frac{d^2t}{dx^2}dx$$

Using eqn (2), we have

$$Q(x+dx) = -kA\frac{dt}{dx} - kA\frac{d^2t}{dx^2}dx \tag{3}$$

Heat convicted out of the element between the planes x and $(x+dx)$.

$$q(c) = h(\rho dx)dt. \tag{4}$$

Now, putting these values in equation (1), we get

$$kA\frac{d^2t}{dx^2}dx - h(\rho dx)dt = 0$$
$$\frac{d^2t}{dx^2}dx - \frac{h(\rho dx)}{kA}dt = 0$$
$$\frac{d^2t}{dx^2}dx - \frac{h(\rho dx)}{kA}\theta = 0$$
$$\text{where } dt = T - T_0 = \theta$$
$$\frac{d^2t}{dx^2}dx - \frac{h(\rho dx)}{kA}\theta = 0$$
$$\text{Where, } m^2 = \frac{h(\rho dx)}{kA}$$
$$\frac{d^2t}{dx^2} - m^2\theta = 0 \tag{4}$$

This is solid fins equation.

3. Mittag-Leffler Function:

The importance of this M-L Function is realized during the last one and a half decades due to its direct involvement in the problems of physics, biology, engineering and applied sciences. Mittag-Leffler function naturally occurs as the solution of fractional order differential or fractional order integral equation. The Mittag-Leffler function is defined and studied by Mittag-Leffler [17] in1903, it is a direct generalization of the exponential function.

$$E_\alpha(z) = \sum_{k=o}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)} \tag{5}$$

$$E_1(z) = \exp(z)$$

And its generalized form which is given below studied by Wiman 1905.

$$E_{\alpha,\beta}(z) = \sum_{k=o}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \alpha,\beta \in C, Re(\alpha) > 0 \tag{6}$$

Generalized Mittag-Leffler function $E_{\alpha,\beta}^{\gamma}(z)$ Was introduced by Parbhakar [24] in 1971 which is generalization of above two parameter Mittag-Leffler function is defined in terms of the series representation as below

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=o}^{\infty}\frac{(\gamma)_k z^k}{\Gamma(\alpha k+\beta)}, \alpha,\beta,\gamma \in C, Re(\alpha) > 0, Re(\beta) > 0 \tag{7}$$

Where $(\gamma)_k$is Pochammer's symbol defined by

$$(\gamma)_k = \begin{cases} 1, k = 0 \\ \gamma(\gamma+1)\dots(\gamma+k-1), k \in N \end{cases}; \gamma \neq 0 \tag{8}$$

It is an entire function of order $\rho = [Re(\alpha)]^{-1}$[24]. This function has been studied by Agarwal [18] and several others. Some special cases of (7) are given below:

$$E_\alpha(z) = E_{\alpha,1}^{1}(z) \tag{9}$$
$$E_{\alpha,\beta}(z) = E_{\alpha,\beta}^{1}(z) \tag{10}$$
$$\phi(\beta,\gamma;z) = {}_1F_1(\beta,\gamma;z)\,\Gamma(\gamma)E_{\alpha,\gamma}^{\beta}(z) \tag{11}$$

where $\phi(\beta,\gamma;z)$ is Kummer's confluent hypergeometric function defined in Erdelyi et al. ([2], p. 248, eq.1]). Mellin-Barnes integral representation for the function defined by (7) follows from the integral

$$E_{\alpha,\beta}^{\gamma}(z) = \frac{1}{2\pi\omega\Gamma(\gamma)}\int_{\Omega}\frac{\Gamma(-s)\Gamma(\gamma+s)(-z)^s}{\Gamma(\beta+\alpha s)}ds \tag{12}$$

where $\omega = (-1)^{1/2}$. The contour is a straight line parallel to the imaginary axis separating the poles of $\Gamma(-s)$ at the points $s = \nu$, $(\nu = 0,1,2,\dots)$ from those of $\Gamma(\gamma+s)$ at the points $s = -\gamma-\nu$ $(\nu = 0,1,2,\dots)$. The poles of the integrand of (12) are assumed to be simple. Eqn. (12) can be established by calculating the residues at the poles of $\Gamma(-s)$ at the points, $s = v$, $(v = 0,1,2,\ldots)$. It follows from (12) that $E_{\alpha,\beta}^{\gamma}(z)$ can be represented in the form

$$E_{\alpha,\beta}^{\gamma}(z) = \frac{1}{\Gamma(\gamma)} H_{1,2}^{1,1}\left[-z\left|\begin{matrix}(1-\gamma,1)\\(0,1),(1-\beta,\alpha)\end{matrix}\right.\right], \alpha,\beta,\gamma \in C, Re(\alpha) > 0, \quad (13)$$

where $H_{1,2}^{1,1}(z)$ is the H-function. A detailed account of the theory and applications of the H-function is available from Mathai and Saxena [15]. This function can also be represented by

$$E_{\alpha,\beta}^{\gamma}(z) = \frac{1}{\Gamma(\gamma)}\, {}_1\psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix};z\right] \quad (14)$$

wher ${}_1\psi_1(z)$ is a special case ofWright ' s generalized hypergeometric function ${}_p\psi_q(z)$ [9] ; also see, Erdelyi et al. ([2-4], Section 4.1), defined by

$${}_p\psi_q\left[\begin{matrix}(a_1,A_1)\ldots(a_p,A_p)\\(b_1,B_1)\ldots(b_q,B_q)\end{matrix};z\right] = \sum_{k=o}^{\infty}\frac{\Pi_{j=1}^{p}\Gamma(a_j + A_j k)}{\Pi_{j=1}^{q}\Gamma(b_j + B_j k)}\frac{z^k}{k!} \quad (15)$$

Where, $$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0,$$

(equality only holds for appropriately bounded z). When $\gamma = 1$, (13) and (14) give rise to (16) and (17) given below:

$$E_{\alpha,\beta}(z) = {}_1\psi_1\left[\begin{matrix}(1,1)\\(\beta,\alpha)\end{matrix};z\right] \quad (16)$$

$$= H_{1,2}^{1,1}\left[-z\left|\begin{matrix}(0,1)\\(0,1),(1-\beta,\alpha)\end{matrix}\right.\right] \quad (17)$$

Where $\alpha,\beta,\gamma \in C, Re(\alpha) > 0$.
If we further take $\beta = 1$ in (16) and (17) we find that

$$E_{\alpha}(z) = {}_1\psi_1\left[\begin{matrix}(1,1)\\(1,\alpha)\end{matrix};z\right] \quad (18)$$

$$= H_{1,2}^{1,1}\left[-z\left|\begin{matrix}(0,1)\\(0,1),(0,\alpha)\end{matrix}\right.\right] \quad (19)$$

Where $\alpha \in C, Re(\alpha) > 0$.

4. Fractional order calculus:

Fractional order calculus is a natural extension of classical mathematics it deals with integrals and derivatives of arbitrary (i.e. non-integer) order it has been investigated mainly from a mathematical point of view. Many mathematicians like Liouville, Lacroix, Laurent, Cauchy, Caputo, [16, 23] have been given different definitions of fractional differentiation and integration. Although all these definition may be equivalent from one specific stand point i.e. for a specific application some definitions seem more attractive.

The definition of fractional integrals and differential used in the analysis are defined below. The commonly used the definition of fractional differentials and integrals due to Reimann-Liouville [27] fractional integral and differential of order α is defined by

$${}_0D_t^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}f(u)du, \quad R(\alpha) > 0, \quad (20)$$

$${}_0D_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^t (t-u)^{n-\alpha-1}f(u)du, \quad R(\alpha) > 0, \quad (21)$$

5. Laplace Transform of fractional order integral and fractional order differential:

The formula (20) is of convolution type, then its Laplace Transform [8, 24] is given by

$$L\{{}_0D_t^{-\alpha}f(t)\} = \frac{1}{\Gamma(\alpha)}L[(t^{\alpha-1}) * f(t)] = \frac{1}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{s^{\alpha}}F(s) = \frac{F(s)}{s^{\alpha}} \quad (22)$$

Where $F(s)$ is the Laplace Transform of (t) .

The Laplace Transform of the fractional derivative of α order is given by Lokenath, [8] 2003.

$$L\{{}_0D_t^{\alpha}f(t)\} = s^{\alpha}F(s) - \sum_{k=o}^{n-1} s^k\, {}_0D_t^{\alpha-k-1}f(0) \quad (23)$$

$$= s^{\alpha}F(s) - \sum_{k=o}^{n-1} s^k C_k \quad (24)$$

where, $(n-1) < \alpha \leq n$ and $C_k = {}_0D_t^{\alpha-k-1}f(0)$

6. Fractional Differential Fin Equation:

Now, we generalized the above fins equation (4) into fractional differential equation as:

$$\frac{d^{\alpha}t}{dx^{\alpha}} - m^{2\alpha-2}\theta = 0 \quad (25)$$

Taking Laplace transform on both sides,

$$s^{\alpha}\theta(s) - \sum_{k=0}^{n-1} s^k D_t^{\alpha-k-1}\theta(0) - m^{2\alpha-2}\theta(s) = 0$$

$$(s^{\alpha} - m^{2\alpha-2})\theta(s) = \sum_{k=0}^{n-1} s^k C_k, \quad \text{Where, } D_t^{\alpha-k-1}\theta(0) = C_k$$

$$\theta(s) = \sum_{k=0}^{n-1} s^k C_k\, s^{-\alpha}(1 - s^{-\alpha}m^{2\alpha-2})^{-1}$$

$$\theta(s) = \sum_{k=0}^{n-1} s^{k-\alpha} C_k \sum_{k=0}^{\infty} \frac{(1)_k}{k!} s^{-\alpha k} m^{(2\alpha-2)k}$$

$$\theta(s) = \sum_{k=0}^{\infty} C_k\, m^{(2\alpha-2)k} s^{-\alpha k+k-\alpha} \quad (26)$$

Taking inverse Laplace transform of both sides, we have

$$\theta(t) = \sum_{k=0}^{\infty} C_k\, m^{(2\alpha-2)k} \frac{t^{\alpha k-k+\alpha-1}}{\Gamma(\alpha k - k + \alpha)} \quad (27)$$

$$\theta(t) = C_k m^{(2\alpha-2)k} t^{\alpha-1} E_{\alpha,\alpha-k}(t^{\alpha-1}) \quad (28)$$

where $E_{\alpha,\alpha-k}(t^{\alpha-1})$ is Mittag – Lef ler function of two parameters. that is normally expected from the fractional differential equation that the solution comes in the form of Mittag-Leffler function.

7. DISCUSSION AND CONCLUSIONS:

Recently, progress in the area of fractional calculus employee a promising potential for future development and application of the theory in various scientific areas the treatment of fractional order calculus in this paper is suggestive rather then rigorous in order to capture the readers interest while simultaneously of ring a hint of its potential as a research tool this article presented a case study involving the implementation of fractional order based model whose results demonstrate the importance of fractional calculus. We strongly hope it will serve as motivation for the development of new applications. The fractional fins equation (25) has been extended to generalized fin equation (4).The solution of fractional fin equation in terms of the ordinary Mittag-Leffler function and their generalization which can also be represented as Fox's H-function. The single parameter Mittag-Leffler function and two parameter Mittag-Leffler functions interpolate between a purely exponential law and power-like behavior of phenomena governed by ordinary fin equation and their fractional counter parts respectively. A specific example of such behavior is the application of Tsallis statistics to phenomena that may arise from fluctuation of temperature or energy dissipation rate (Lavagno and quarati, 2002) the application of fin fractional fin equation to describe such phenomena has not been fully developed yet.

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