

Triple Dirichlet Average of Hyperbolic Functions and Fractional Derivative

Mohd. Farman Ali¹, Renu Jain², Manoj Sharma³

^{1,2} School of Mathematics and Allied Sciences, Jiwaji University, Gwalior

³ Department of Mathematics RJIT, BSF Academy, Tekanpur

Abstract - The aim of present paper to establish some results of Triple Dirichlet average of hyperbolic functions i.e. $\cosh x$ and $\sinh x$, using fractional derivative.

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1. INTRODUCTION:

Carlson [1-5] has defined Dirichlet average of functions which represents certain type of integral average with respect to Dirichlet measure. He showed that various important special functions can be derived as Dirichlet averages for the ordinary simple functions like x^t, e^x etc. He has also pointed out [3] that the hidden symmetry of all special functions which provided their various transformations can be obtained by averaging x^n, e^x etc. Thus he established a unique process towards the unification of special functions by averaging a limited number of ordinary functions. Almost all known special functions and their well known properties have been derived by this process. Recently, Gupta and Agarwal [10,11] found that averaging process is not altogether new but directly connected with the old theory of fractional derivative. Carlson overlooked this connection whereas he has applied fractional derivative in so many cases during his entire work. Deora and Banerji [6] have found the double Dirichlet average of e^x by using fractional derivatives and they have also found the Triple Dirichlet Average of x^t by using fractional derivatives [8]. We can say that every analytic functions can be measured as Dirichlet average, using fractional derivative. In the present paper the Dirichlet average of hyperbolic functions has been obtained.

2. DEFINITIONS:

We give below some of the definitions which are necessary in the preparation of this paper.

2.1 STANDARD SIMPLEX IN $R^n, n \geq 1$:

We denote the standard simplex in $R^n, n \geq 1$ by [1, p.62].

$$E = E_n = \{S(u_1, u_2, \dots, u_n) : u_1 \geq 0, \dots, u_n \geq 0, u_1 + u_2 + \dots + u_n \leq 1\}$$

2.2 DIRICHLET MEASURE:

Let $b \in C^k, k \geq 2$ and let $E = E_{k-1}$ be the standard simplex in R^{k-1} . The complex measure μ_b is defined by $E[1]$.

$$d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1-1} \dots u_{k-1}^{b_{k-1}-1} (1 - u_1 - \dots - u_{k-1})^{b_k-1} du_1 \dots du_{k-1} \quad (2.2.1)$$

Will be called a Dirichlet measure.

Here

$$B(b) = B(b_1, \dots, b_k) = \frac{\Gamma(b_1) \dots \Gamma(b_k)}{\Gamma(b_1 + \dots + b_k)},$$

$$C_{>} = \{z \in \mathbb{C} : z \neq 0, |\arg z| < \pi/2\},$$

Open right half plane and $C_{>}^k$ is the k^{th} Cartesian power of $C_{>}$

2.3 DIRICHLET AVERAGE [1, p.75]:

Let Ω be the convex set in $C_{>}^k$, let $z = (z_1, \dots, z_k) \in \Omega^k, k \geq 2$ and let $u.z$ be a convex combination of z_1, \dots, z_k . Let f be a measurable function on Ω and let μ_b be a Dirichlet measure on the standard simplex E in R^{k-1} . Define

$$F(b, z) = \int_E f(u.z) d\mu_b(u) \quad (2.3)$$

We shall call F the Dirichlet measure of f with variables

$z = (z_1, \dots, z_k)$ and parameters $b = (b_1, \dots, b_k)$.

Here

$$u.z = \sum_{i=1}^k u_i z_i \text{ and } u_k = 1 - u_1 - \dots - u_{k-1}.$$

If $k = 1$, define $F(b, z) = f(z)$.

2.4 FRACTIONAL DERIVATIVE [9, P.181]:

The concept of fractional derivative with respect to an arbitrary function has been used by Erdelyi [9]. The most common definition for the fractional derivative of order α found in the literature on the ‘‘Riemann-Liouville integral’’ is

$$D_z^\alpha F(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z F(t)(z-t)^{-\alpha-1} dt \tag{2.4}$$

Where $Re(\alpha) < 0$ and $F(x)$ is the form of $x^p f(x)$, where $f(x)$ is analytic at $x = 0$.

2.5 AVERAGE OF cosh x (FROM [16]):

let μ^b be a Dirichlet measure on the standard simplex E in R^{k-1} ; $k \geq 2$. For every $z \in C^k$

$$S(b, z) = \int_E \cosh(u.z) d\mu_b(u) \tag{2.5}$$

If $k = 1, S = (b, z) = \cosh(u.z)$.

2.6 TRIPLE AVERAGES OF FUNCTIONS OF ONE VARIABLE (FROM [1, 2]): let z be species with complex elements z_{ijk} . Let $u = (u_1, \dots, u_l)$, $v = (v_1, \dots, v_m)$ and $w = (w_1, \dots, w_n)$ be an ordered l-tuple, m-tuple and n-tuple of real non-negative weights $\sum u_i = 1$, $\sum v_j = 1$, and $\sum w_k = 1$ respectively.

We define

$$u.z.v.w = \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n u_i z_{ijk} v_j w_k \tag{2.6}$$

If z_{ijk} is regarded as a point of the complex plane, all these convex combinations are points in the convex hull of (z_{11}, \dots, z_{kn}) , denote by $H(z)$.

Let $\mu = (\mu_1, \dots, \mu_l)$ be an ordered l-tuple of complex numbers with positive real part ($Re(\mu) > 0$) and similarly for $\alpha = (\alpha_1, \dots, \alpha_m)$ and $\beta = (\beta_1, \dots, \beta_n)$. Then we define $dm_\mu(u)$, $dm_\alpha(v)$ and $dm_\beta(w)$ as (2.2.1).

Let f be the holomorphic on a domain D in the complex plane. If $Re(\mu) > 0, Re(\alpha), Re(\beta) > 0$ and $H(z) \subset D$, we define

$$F(\mu, z, \alpha, \beta) = \iiint \int f(u, z, v, w) dm_\mu(u) dm_\alpha(v) dm_\beta(w) \tag{2.7}$$

Corresponding to the particular function $\cosh x, z^t$ and e^z , we define,

$$S(\mu, z, \alpha, \beta) = \iiint \int \cosh(u, z, v, w) dm_\mu(u) dm_\alpha(v) dm_\beta(w) \tag{2.8}$$

$$S(\mu, z, \alpha, \beta) = \iiint \int \sin h(u, z, v, w) dm_\mu(u) dm_\alpha(v) dm_\beta(w) \tag{2.9}$$

$$R_t(\mu, z, \alpha, \beta) = \iiint \int (u, z, v, w)^t dm_\mu(u) dm_\alpha(v) dm_\beta(w) \tag{2.10}$$

$$S(\mu, z, \alpha, \beta) = \iiint \int (e)^{u.z.v.w} dm_\mu(u) dm_\alpha(v) dm_\beta(w) \tag{2.11}$$

3. MAIN RESULTS AND PROOF:

Theorem: Following equivalence relation for Triple Dirichlet Average is established for $(l = m = n = 2)$ of $\cosh(u, z, v, w)$

$$S(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{\Gamma(\mu + \mu')}{\Gamma\mu} (x - y)^{1-\mu-\mu'} D_{x-y}^{-\mu'} \cosh x (x - y)^{\mu-1} \tag{3.1}$$

Proof:

Let us consider the triple average for $(l = m = n = 2)$ of $\cosh(u.z.v.w)$

$$S(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \int_0^1 \int_0^1 \int_0^1 \cosh(u.z.v.w) dm_{\mu, \mu'}(u) dm_{\alpha, \alpha'}(v) dm_{\beta, \beta'}(w)$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \int_0^1 \int_0^1 [u.z.v.w]^{2n} dm_{\mu,\mu'}(u) dm_{\alpha,\alpha'}(v) dm_{\beta,\beta'}(w) \quad (3.2)$$

$Re(\mu) = 0, Re(\mu') = 0, Re(\alpha) > 0, Re(\alpha') > 0, Re(\beta) > 0, Re(\beta') > 0$ and

$$u.z.v.w = \sum_{i=1}^2 \sum_{j=1}^2 \sum_{k=1}^2 u_i z_{ijk} v_j w_k = \sum_{i=1}^2 \sum_{j=1}^2 [u_i v_j (z_{ij1} w_1 + z_{ij2} w_2)]$$

$$u.z.v.w = \sum_{i=1}^2 [u_i (v_1 z_{i11} w_1 + v_1 z_{i12} w_2 + v_2 z_{i21} w_1 + v_2 z_{i22} w_2)]$$

$$u.z.v.w = [u_1 v_1 z_{111} w_1 + u_1 v_1 z_{112} w_2 + u_1 v_2 z_{121} w_1 + u_1 v_2 z_{122} w_2 + u_2 v_1 z_{211} w_1 + u_2 v_1 z_{212} w_2 + u_2 v_2 z_{221} w_1 + u_2 v_2 z_{222} w_2]$$

let in first species $z_{111} = a, z_{112} = b, z_{121} = c, z_{122} = d$ and second species

$$z_{211} = e, z_{212} = f, z_{221} = g, z_{222} = h$$

$$\text{and } \begin{cases} u_1 = u, & u_2 = 1 - u \\ v_1 = v, & v_2 = 1 - v \\ w_1 = w, & w_2 = 1 - w \end{cases}$$

such that

$$u.z.v.w = [uvw(a - b - c + d - e + f + g - h) + uv(b - d - f + h) + vw(e - f - g + h) + wu(c - d - g + h) + u(d - h) + v(f - h) + w(g - h) + h]$$

$$dm_{\mu,\mu'}(u) = \frac{\Gamma(\mu + \mu')}{\Gamma\mu \Gamma\mu'} u^{\mu-1} (1-u)^{\mu'-1} du$$

$$dm_{\alpha,\alpha'}(v) = \frac{\Gamma(\alpha + \alpha')}{\Gamma\alpha \Gamma\alpha'} v^{\alpha-1} (1-v)^{\alpha'-1} dv$$

$$dm_{\beta,\beta'}(w) = \frac{\Gamma(\beta + \beta')}{\Gamma\beta \Gamma\beta'} w^{\beta-1} (1-w)^{\beta'-1} dw$$

Putting these values in (3.2), we have,

$$S(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{\Gamma(\mu + \mu') \Gamma(\rho + \rho') \Gamma(\beta + \beta')}{\Gamma\mu \Gamma\mu' \Gamma\rho \Gamma\rho' \Gamma\beta \Gamma\beta'} \times \sum_{n=0}^{\infty} \frac{1}{(2n)!} \int_0^1 \int_0^1 \int_0^1 [uvw(a - b - c + d - e + f + g - h) + uv(b - d - f + h) + vw(e - f - g + h) + wu(c - d - g + h) + u(d - h) + v(f - h) + w(g - h) + h]^{2n} \times u^{\mu-1} (1-u)^{\mu'-1} du v^{\alpha-1} (1-v)^{\alpha'-1} dv w^{\beta-1} (1-w)^{\beta'-1} dw \quad (3.3)$$

In order to obtained the fractional derivative equivalent to the above integral.

CASE-1:

If $a = x, e = y, b = c = d = f = g = h = 0$ then we have

$$S(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{\Gamma(\mu + \mu') \Gamma(\rho + \rho') \Gamma(\beta + \beta')}{\Gamma\mu \Gamma\mu' \Gamma\rho \Gamma\rho' \Gamma\beta \Gamma\beta'} \times \sum_{n=0}^{\infty} \frac{1}{(2n)!} \int_0^1 \int_0^1 \int_0^1 [uvw(x - y) + vwy]^{2n} u^{\mu-1} (1-u)^{\mu'-1} v^{\alpha-1} (1-v)^{\alpha'-1} w^{\beta-1} (1-w)^{\beta'-1} dudvdw$$

Using the definition of beta function and due to suitable adjustment we arrive at

$$S(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{(\alpha)_n(\beta)_n}{(\alpha + \alpha')_n(\beta + \beta')_n} \frac{\Gamma(\mu + \mu')}{\Gamma\mu \Gamma\mu'} \\ \times \sum_{n=0}^{\infty} \frac{1}{(2n)!} \int_0^1 [ux + (1-u)y]^{2n} u^{\mu-1}(1-u)^{\mu'-1} du \\ S(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{(\alpha)_n(\beta)_n}{(\alpha + \alpha')_n(\beta + \beta')_n} S(\mu, \mu'; x, y) \quad (3.4)$$

By using the definition of fractional derivative we get,

$$S(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{(\alpha)_n(\beta)_n}{(\alpha + \alpha')_n(\beta + \beta')_n} \frac{\Gamma(\mu + \mu')}{\Gamma\mu} (x-y)^{1-\mu-\mu'} D_{x-y}^{-\mu'} \cosh x (x-y)^{\mu-1} \quad (3.5)$$

This is complete proof of (3.1).

Similarly we can show equivalence of triple Dirichlet average of $\sinh x$ ($l = m = n = 2$) with the fractional derivative i.e

$$S(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{(\alpha)_n(\beta)_n}{(\alpha + \alpha')_n(\beta + \beta')_n} \frac{\Gamma(\mu + \mu')}{\Gamma\mu} (x-y)^{1-\mu-\mu'} D_{x-y}^{-\mu'} \sinh x (x-y)^{\mu-1} \quad (3.6)$$

CASE 2:

If we put $a = b = c = d = x; e = f = g = h = y$ in equation (3.3) then Triple Dirichlet average of $\cosh x$ is same as single Dirichlet average of $\cosh x$ Sharma and Jain [14], i.e.

$$S(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{\Gamma(\mu + \mu') \Gamma(\rho + \rho') \Gamma(\beta + \beta')}{\Gamma\mu \Gamma\mu' \Gamma\rho \Gamma\rho' \Gamma\beta \Gamma\beta'} \\ \times \sum_{v=0}^{\infty} \frac{1}{(2n)!} \int_0^1 \int_0^1 \int_0^1 [u(x-y) + y]^{2n} u^{\mu-1}(1-u)^{\mu'-1} du v^{\alpha-1}(1-v)^{\alpha'-1} dv w^{\beta-1}(1-w)^{\beta'-1} dw \\ S(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{\Gamma(\mu + \mu')}{\Gamma\mu \Gamma\mu'} \times \sum_{n=0}^{\infty} \frac{1}{(2n)!} \int_0^1 [u(x-y) + y]^{2n} u^{\mu-1}(1-u)^{\mu'-1} du$$

Putting $u(x-y) = t$, we obtain

$$S(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{\Gamma(\mu + \mu')}{\Gamma\mu \Gamma\mu'} \times \sum_{n=0}^{\infty} \frac{1}{(2n)!} \int_0^{x-y} [y+t]^{2n} \left(\frac{t}{x-y}\right)^{\mu-1} \left(1 - \frac{t}{x-y}\right)^{\mu'-1} \frac{dt}{(x-y)}$$

$$S(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{\Gamma(\mu + \mu')}{\Gamma\mu \Gamma\mu'} (x-y)^{1-\mu-\mu'} \sum_{n=0}^{\infty} \frac{1}{(2n)!} \int_0^{x-y} [y+t]^{2n} (t)^{\mu-1} (x-y-t)^{\mu'-1} dt$$

On changing the order of integration and summation, we have

$$S(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{\Gamma(\mu + \mu')}{\Gamma\mu \Gamma\mu'} (x-y)^{1-\mu-\mu'} \int_0^{x-y} \cosh(y+t) (t)^{\mu-1} (x-y-t)^{\mu'-1} dt \quad (3.7)$$

Using definition of fractional derivative (2.4), we get

$$S(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{\Gamma(\mu + \mu')}{\Gamma\mu} (x-y)^{1-\mu-\mu'} D_{x-y}^{-\mu'} \cosh x (x-y)^{\mu-1}$$

Hence we obtained Triple Dirichlet average of $\cosh x$ is same as single Dirichlet average of $\cosh x$.

Similarly we can show equivalence of triple Dirichlet average of $\sinh x$ ($l = m = n = 2$) with the fractional derivative i.e.

$$S(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{\Gamma(\mu + \mu')}{\Gamma\mu} (x-y)^{1-\mu-\mu'} D_{x-y}^{-\mu'} \sinh x (x-y)^{\mu-1} \quad (3.8)$$

4. PARTICULAR CASES:

- (i) If $\mu' = v - \mu$ and $y = 0$ in (3.1)

$$S(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{(\alpha)_n(\beta)_n}{(\alpha + \alpha')_n(\beta + \beta')_n} \left(\frac{\Gamma v}{\Gamma\mu}\right) (x)^{1-v} D_x^{\mu-v} \cosh x (x)^{\mu-1}$$

$$S(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{(\alpha)_n(\beta)_n}{(\alpha + \alpha')_n(\beta + \beta')_n} \frac{1}{2} \left[\frac{\Gamma v}{\Gamma \mu} (x)^{1-v} D_x^{\mu-v} e^x (x)^{\mu-1} + \frac{\Gamma v}{\Gamma \mu} (x)^{1-v} D_x^{\mu-v} e^{-x} (x)^{\mu-1} \right]$$

$$S(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \frac{(\alpha)_n(\beta)_n}{(\alpha + \alpha')_n(\beta + \beta')_n} \frac{1}{2} [{}_1F_1(\mu, v; x) + {}_1F_1(\mu, v; -x)] \quad (4.1)$$

(ii) If $\mu = -n, \mu' = 1 + \gamma + n$ and $y = 0$ in (3.1) we have,

$$S(-n, 1 + \gamma + n; z; \alpha, \alpha', \beta, \beta') = \frac{(\alpha)_n(\beta)_n}{(\alpha + \alpha')_n(\beta + \beta')_n} \frac{1}{2} \left[\frac{\Gamma(1 + \gamma)}{\Gamma(-n)} x^{-\gamma} D_x^{-n-\gamma-1} e^x x^{-n-1} + \frac{\Gamma(1 + \gamma)}{\Gamma(-n)} x^{-\gamma} D_x^{-n-\gamma-1} e^{-x} x^{-n-1} \right] \quad (4.2)$$

$$S(-n, 1 + \gamma + n; z; \alpha, \alpha', \beta, \beta') = \frac{(\alpha)_n(\beta)_n}{(\alpha + \alpha')_n(\beta + \beta')_n} \frac{1}{2} [{}_1F_1(-n, 1 + \gamma; x) + {}_1F_1(-n, 1 + \gamma; -x)] \quad (4.3)$$

$$S(-n, 1 + \gamma + n; z; \alpha, \alpha', \beta, \beta') = \frac{1}{2} \frac{(\alpha)_n(\beta)_n}{(\alpha + \alpha')_n(\beta + \beta')_n} \left[\frac{L_n^Y(x)}{L_n^Y(0)} + \frac{L_n^Y(-x)}{L_n^Y(0)} \right] \quad (4.4)$$

Where L_n^Y is the Laguerre polynomial of degree n.

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REFERENCES:

- [1] Carlson, B.C., Special Function of Applied Mathematics, Academic Press, New York, 1977.
- [2] Carlson, B.C., Appell's function F_4 as a double average, SIAM J.Math. Anal.6 (1975), 960-965.
- [3] Carlson, B.C., Hidden symmetries of special functions, SIAM Rev. 12 (1970), 332-345.
- [4] Carlson, B.C., Dirichlet averages of $x^l \log x$, SIAM J.Math. Anal. 18(2) (1987), 550-565.
- [5] Carlson, B.C., A connection between elementary functions and higher transcendental functions, SIAM J. Appl. Math. 17 (1969), 116-140.
- [6] Deora, Y. and Banerji, P.K., Double Dirichlet average of e^x using fractional derivatives, J. Fractional Calculus 3 (1993), 81-86.
- [7] Deora, Y. and Banerji, P.K., Double Dirichlet average and fractional derivatives, Rev.Tec.Ing.Univ. Zulia 16(2) (1993), 157-161.
- [8] Deora, Y. and Banerji, P.K., Triple Dirichlet average and fractional derivatives, Rev.Tec.Ing.Univ. Zulia 16(2) (1993), 157-161
- [9] Erdelyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G., Tables of Integral Transforms, Vol.2 McGraw-Hill, New York, 1954.
- [10] Gupta, S.C. and Agrawal, B.M., Dirichlet average and fractional derivatives, J. Indian Acad.Math. 12(1) (1990), 103-115.
- [11] Gupta, S.C. and Agrawal, Double Dirichlet average of e^x using fractional derivatives, Ganita Sandesh 5 (1) (1991), 47-52.
- [12] Kiryakova V., some special functions related to fractional calculus and fractional (non-integer) order control systems and equations. Facta Universitatis (Sci. J. of Univ. Nis) Automatic Control and Robotics, 7 No.1 (2008), 79-98.
- [13] Mathai, A.M. and Saxena, R.K., The H-Function with Applications in Statistics and other Disciplines, Wiley Halsted, New York, 1978.
- [14] Samko, S., Kilbas, A. Marichev, O. Fractional Integrals and derivatives, Theory and Applications. Gordon and Breach, New York (1993).
- [15] Sharma, Manoj and Jain, Renu, Dirichlet Average and Fractional Derivative, J. Indian Acad. Math. Vol.12, No. 1(1990).
- [16] Sharma, Manoj and Jain, Renu, Dirichlet Average of $\cosh x$ and Fractional Derivative, J. Indian Acad. Math. Vol.2, No. 1(2007). P17-22.