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Dirichlet Average of Hyper-geometric Function and Fractional Derivative

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Abstract: In this paper we establish a relation Dirichlet average of Hyper-geometric function, using fractional derivative.

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1. INTRODUCTION:

Carlson [1-5] has defined Dirichlet average of functions which represents certain type of integral average with respect to Dirichlet measure. He showed that various important special functions can be derived as Dirichlet averages for the ordinary simple functions likex^t,e^x etc. He has also pointed out [3] that the hidden symmetry of all special functions which provided their various transformations can be obtained by averaging x^n,e^x etc. Thus he established a unique process towards the unification of special functions by averaging a limited number of ordinary functions. Almost all known special functions and their well known properties have been derived by this process. In this paper the Dirichlet average of Hyper-geometric function has been obtained.

2. DEFINITIONS:

We give blew some of the definitions which are necessary in the preparation of this paper.

2.1 STANDARD SIMPLEX IN $R^n, n \geq 1$:

We denote the standard simplex in R^n, $n \geq 1$ by [1, p.62].

$$E = E_n = \{S(u_1, u_2, \dots \dots . u_n) \;:\; u_1 \geq 0, \dots \dots . u_n \geq 0,\; u_1 + u_2 + \cdots \dots \dots + u_n \leq 1\} \tag{2.1.1}$$

2.2 DIRICHLET MEASURE:

Let $b \in C^k, k \geq 2$ and let $E = E_{k-1}$ be the standard simplex in R^{k-1}. The complex measure μ_b is defined by E[1].

$$d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1-1} \dots \dots \dots \dots . u_{k-1}^{b_{k-1}-1} (1 - u_1 - \cdots \dots \dots - u_{k-1}) b_k^{-1} du_1 \dots \dots \dots . du_{k-1} \tag{2.2.1}$$

Will be called a Dirichlet measure.
Here

$$B(b) = B(b1, \dots \dots . bk) = \frac{\Gamma(b_1) \dots \dots \dots \dots . \Gamma(b_k)}{\Gamma(b_1 + \cdots \dots \dots . + b_k)},$$

$$C_> = \{z \in z : z \neq 0, |ph\, z| < \pi/2\},$$

Open right half plane and $C_>$k is the k^{th} Cartesian power of $C_>$

2.3 DIRICHLET AVERAGE[1, P.75]:

Let Ω be the convex set in $C_>$, let $z = (z_1, \dots \dots \dots , z_k) \in \Omega^k$, $k \geq 2$ and let $u.z$ be a convex combination of $z_1, \dots \dots \dots , z_k$. Let f be a measureable function on Ω and let μ_b be a Dirichlet measure on the standard simplex E in R^{k-1}.Define

$$F(b, z) = \int_E f(u.z) d\,\mu_b(u) \tag{2.3.1}$$

We shall call F the Dirichlet measure of f with variables $z = (z_1, \dots \dots \dots , z_k)$ and parameters $b = (b_1, \dots \dots \dots . b_k)$.
Here

$$u.z = \sum_{i=1}^{k} u_i z_i \text{ and } u_k = 1 - u_1 - \cdots \dots \dots . - u_{k-1} \tag{2.3.2}$$

If $k = 1$, define $F(b, z) = f(z)$.

2.4 FRACTIONAL DERIVATIVE [8, P.181]:

The concept of fractional derivative with respect to an arbitrary function has been used by Erdelyi[8]. The most common definition for the fractional derivative of order α found in the literature on the "Riemann-Liouville integral" is

$$D_z^{\alpha} F(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z F(t)(z-t)^{-\alpha-1} dt \qquad (2.4.1)$$

Where $Re(\alpha) < 0$ and $F(x)$ is the form of $x^p f(x)$, where $f(x)$ is analytic at $x = 0$.

2.5 HYPERGEOMETRIC FUNCTION:

We defined the Hypergeometric function

$${}_pF_q\ (a_1 \ldots . a_p;, b_1 \ldots . b_q; z\ ) = \sum_{n=o}^{\infty} \frac{(a_1)_n \ldots\ldots (a_p)_n}{(b_1)_n \ldots\ldots (b_q)_n} \frac{z^n}{n!} \qquad (2.5.1)$$

Here, α ϵ C, $R(\alpha) > 0$, $(a_j)_k\ (b_j)_k$ are pochammer symbols.

3. EQUIVALENCE:

In this section we shall show the equivalence of single Dirichlet average of Hypergeometric function $(k = 2)$ with the fractional derivative i.e.

$$S(\beta, \beta'; x, y) = \frac{\Gamma(\beta+\beta')}{\Gamma\beta} (x-y)^{1-\beta-\beta'} D_{x-y}^{-\beta'} {}_pF_q(x)(x-y)^{\beta-1} \qquad (3.1)$$

Proof:

$$S(\beta, \beta'; x, y) = \sum_{n=o}^{\infty} \frac{(a_1)_n \ldots\ldots (a_p)_n}{(b_1)_n \ldots\ldots (b_q)_n} \frac{1}{n!} R_n(\beta, \beta'; x, y)$$

$$= \sum_{n=o}^{\infty} \frac{(a_1)_n \ldots\ldots (a_p)_n}{(b_1)_n \ldots\ldots (b_q)_n} \frac{1}{n!} \frac{\Gamma(\beta+\beta')}{\Gamma\beta\, \Gamma\beta'} \int_0^1 [ux + (1-u)y]^n\, u^{\beta-1}(1-u)^{\beta'-1} du$$

Putting $u(x-y) = t$, we have,

$$= \sum_{n=o}^{\infty} \frac{(a_1)_n \ldots\ldots (a_p)_n}{(b_1)_n \ldots\ldots (b_q)_n} \frac{1}{n!} \frac{\Gamma(\beta+\beta')}{\Gamma\beta\, \Gamma\beta'} \int_0^{x-y} [t+y]^n \left(\frac{t}{x-y}\right)^{\beta-1} \left(1 - \frac{t}{x-y}\right)^{\beta'-1} \frac{dt}{x-y}$$

On changing the order of integration and summation, we have

$$= (x-y)^{1-\beta-\beta'} \frac{\Gamma(\beta+\beta')}{\Gamma\beta\, \Gamma\beta'} \int_0^{x-y} \sum_{n=o}^{\infty} \frac{(a_1)_n \ldots\ldots (a_p)_n}{(b_1)_n \ldots\ldots (b_q)_n} \frac{1}{n!} [t+y]^n\, (t)^{\beta-1}(x-y-t)^{\beta'-1} dt$$

Or

$$= (x-y)^{1-\beta-\beta'} \frac{\Gamma(\beta+\beta')}{\Gamma\beta\, \Gamma\beta'} \int_0^{x-y} {}_pF_q(y+t)\, (t)^{\beta-1}(x-y-t)^{\beta'-1} dt$$

Hence by the definition of fractional derivative, we get

$$S(\beta, \beta'; x, y) = (x-y)^{1-\beta-\beta'} \frac{\Gamma(\beta+\beta')}{\Gamma\beta} D_{x-y}^{-\beta'} {}_pF_q(x)(x-y)^{\beta-1}$$

This completes the Analysis.

4. PARTICULAR CASES:

(4.1). If $\beta' = \gamma - \beta,\ \ y = 0$ and no upper and lower parameter in (3.1) then

$$S(\beta, \gamma-\beta; x, 0) = (x)^{1-\gamma} \frac{\Gamma\gamma}{\Gamma\beta} D_x^{\beta-\gamma} e^x (x)^{\beta-1}$$

$$= {}_1F_1(\beta; \gamma; x) \qquad (4.1)$$

This confluent hyper geometric function. [11]

$$S(\beta, \gamma-\beta; x, 0) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma s \frac{\Gamma(\beta-s)}{\Gamma(\gamma-s)} (-x)^{-s} ds \qquad (4.2)$$

Then

$$S(\beta,\gamma-\beta;x,0)=\frac{\Gamma(\gamma)}{\Gamma(\beta)}\left[H_{1,2}^{1,1}\left(-x\left|\begin{matrix}(1-\beta,1)\\(0,1),(1-\gamma,1)\end{matrix}\right.\right)\right] \tag{4.3}$$

(ii) If $\beta' = \xi - \beta$ and From (4.1), then

$$S(\beta,\xi-\beta;x,0)=\frac{\Gamma(\xi)}{\Gamma(\beta)}x^{1-\xi}D_x^{\beta-\xi}e^x x^{\beta-1} \tag{4.4}$$

$$S(\beta,\xi-\beta;x,0)={}_1F_1(\beta,\xi;x)=\Gamma(\xi)E_{1,\xi}^{\beta}(x) \tag{4.5}$$

Where $E_{1,\xi}^{\beta}(x)$ be the generalization of Mittag-Leffler function [12].

(iii) If $\beta = -n, \beta' = 1+\alpha+n, \gamma = 0$ and no upper and lower parameter in (3.1) then

$$S(-n,1+\alpha+n;\ x,0)=(x)^{-\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(-n)}D_x^{-n-\alpha-1}e^x(x)^{-n-1}$$

$$={}_1F_1(-n,1+\alpha;x)$$

$$=\frac{L_n^{\alpha}(x)}{L_n^{\alpha}(0)} \tag{4.6}$$

Where $L_n^{\alpha}(x)$ is the Laguerre polynomial of degree n.

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