Generalized Yang-Fourier Transforms to Heat-Conduction in a Semi-Infinite Fractal Bar

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Abstract: The aim of present paper to solve 1-D fractal heat-conduction problem in a fractal semi-infinite bar has been developed by local fractional calculus employing the analytical Generalized Yang-Fourier transforms method.

Keywords: fractal bar, heat-conduction equation, Generalized Yang-Fourier transforms, Yang-Fourier transforms, local fractional calculus.

1. Introduction:

Generalized Yang-Fourier transforms which is obtained by authors by generalization of Yang-Fourier transforms is a technique of fractional calculus for solving mathematical, physical and engineering problems. The fractional calculus is continuously growing in last five decades [1-7]. Most of the fractional ordinary differential equations have exact analytic solutions, while others required either analytical approximations or numerical techniques to be applied, among them: fractional Fourier and Laplace transforms [8,41], heat-balance integral method [9-11], variation iteration method (VIM) [12-14], decomposition method [15,41], homotopy perturbation method [16,41] etc.

The problems in fractal media can be successfully solved by local fractional calculus theory with problems for non-differential functions [25-32]. Local fractional differential equations have been applied to model complex systems of fractal physical phenomena [30-41] local fractional Fourier series method [38], Yang-Fourier transform [39, 40, 41]

2. Generalized Yang-Fourier transform and its properties:

Let us consider \( f(x) \) is local fractional continuous in \((-\infty, \infty)\) we denote as \( f(x) \in C_{a,b}(-\infty, \infty) \) [32, 33, 35].

Let \( f(x) \in C_{a,b}(-\infty, \infty) \) The Generalized Yang-Fourier transform developed by authors is written in the form [30, 31, 39, 40, 41]:

\[
F_{a,b} \{f(x)\} = f^{F \alpha \beta}_{\omega} (\omega) = \frac{1}{\Gamma(1+\alpha+\beta)} \int_{-\infty}^{\infty} E_{a,b} \left[-i^{a+b} \omega^{a+b} x^{a+b}\right] f(x) (dx)^{a+b} \tag{1}
\]

When we put \( \beta \) equal to zero, it converts in to the Yang-Fourier transform [41].

Then, the local fractional integration is given by [30-32, 35-37, 41]:

\[
\frac{1}{\Gamma(1+\alpha+\beta)} \int_{a}^{b} f(t) (dx)^{a+b} = \frac{1}{\Gamma(1+\alpha+\beta)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_{j}) (\Delta t)^{a+b} \tag{2}
\]

where \( \Delta t_{j} = t_{j+1} - t_{j}, \Delta t = \max\{\Delta t_{1}, \Delta t_{2}, \Delta t_{j}, \ldots\} \) and \( \{t_{j}, t_{j+1}\}, j = 0, \ldots, N-1, t_{0} = a, t_{N} = b, \) is a partition of the interval \([a, b]\).

If \( F_{a,b} \{f(x)\} = f^{F \alpha \beta}_{\omega} (\omega) \), then its inversion formula takes the form [30, 31, 39, 40, 41]

\[
f(x) = F^{-1}_{a,b} \left[f^{F \alpha \beta}_{\omega} (\omega)\right] = \frac{1}{(2\pi)^{a+b}} \int_{-\infty}^{\infty} E_{a,b} \left[-i^{a+b} \omega^{a+b} x^{a+b}\right] f^{F \alpha \beta}_{\omega} (\omega) (d\omega)^{a+b} \tag{3}
\]

When we put \( \beta \) equal to zero, it converts in to the Yang Inverse Fourier transform [41].

Some properties are shown as it follows [30, 31]:
Let \( F_{\alpha, \beta} \{ f(x) \} = f_0^{F_{\alpha, \beta}}(\omega) \), and \( F_{\alpha, \beta} \{ g(x) \} = f_0^{F_{\alpha, \beta}}(\omega) \), and let be two constants. Then we have:

\[
F_{\alpha, \beta} \{ cf'(x) + dg(x) \} = cF_{\alpha, \beta} \{ f(x) \} + dF_{\alpha, \beta} \{ g(x) \}
\]

(4)

If \( \lim_{|x| \to \infty} f(x) = 0 \), then we have:

\[
F_{\alpha, \beta} \{ f^{\alpha+\beta}(x) \} = t^{\alpha+\beta} \omega^{\alpha+\beta} F_{\alpha, \beta} \{ f(x) \}
\]

(5)

In eq. (5) the local fractional derivative is defined as:

\[
f^{\alpha+\beta}(x_0) = \frac{d^{\alpha+\beta} f(x)}{dx^{\alpha+\beta}} \bigg|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha+\beta} [f(x) - f(x_0)]}{(x - x_0)^{\alpha+\beta}}
\]

(6)

Where \( \Delta^{\alpha+\beta} [f(x) - f(x_0)] \equiv \Gamma(1 + \alpha + \beta) \Delta [f(x) - f(x_0)] \)

As a direct result, repeating this process, when:

\[
f(0) = f^{(k)}(0) = \cdots = f^{(k-1)}(0) = 0
\]

(7)

\[
F_{\alpha, \beta} \{ f^{(k+1)}(x) \} = t^{\alpha+\beta} \omega^{\alpha+\beta} F_{\alpha, \beta} \{ f(x) \}
\]

(8)

3. Heat conduction in a fractal semi-infinite bar:

If a fractal body is subjected to a boundary perturbation, then the heat diffuses in depth modeled by a constitutive relation where the rate of fractal heat flux \( \overline{q}(x,y,z,t) \) is proportional to the local fractional gradient of the temperature [32,41], namely:

\[
\overline{q}(x,y,z,t) = -K^{\alpha+\beta} \overline{q}^{\alpha+\beta} T(x,y,z,t)
\]

(9)

Here the pre-factor \( K^{\alpha+\beta} \) is the thermal conductivity of the fractal material. Therefore, the fractal heat conduction equation without heat generation was suggested in [32] as:

\[
K^{2+\alpha+\beta} \frac{d^{2+\alpha+\beta} T(x,y,z,t)}{dx^{2+\alpha+\beta}} - \rho^{\alpha+\beta} c^{\alpha+\beta} \frac{d^{2+\alpha+\beta} T(x,y,z,t)}{dx^{2+\alpha+\beta}} = 0
\]

(10)

Where \( \rho^{\alpha+\beta} \) and \( c^{\alpha+\beta} \) are the density and the specific heat of material, respectively.

The fractal heat-conduction equation with a volumetric heat generation \( g(x,y,z,t) \) can be described as [32,41]:

\[
K^{2+2\alpha+2\beta} \overline{q}^{2+2\alpha+2\beta} T(x,y,z,t) + g(x,y,z,t) \rho^{\alpha+\beta} c^{\alpha+\beta} \frac{\partial^{(\alpha+\beta)} T(x,y,z,t)}{\partial t^{(\alpha+\beta)}}
\]

(11)

The 1-D fractal heat-conduction equation [32,41] reads as:

\[
K^{2+\alpha+2\beta} \frac{d^{2+\alpha+2\beta} T(x,t)}{dx^{2+\alpha+2\beta}} - \rho^{\alpha+\beta} c^{\alpha+\beta} \frac{d^{2+\alpha+2\beta} T(x,t)}{dx^{2+\alpha+2\beta}} = 0, \quad 0 < x < \infty, t > 0
\]

(12a)

with initial and boundary conditions:

\[
\frac{\partial^{(\alpha+\beta)} T(0,t)}{\partial t^{(\alpha+\beta)}} = E^{\alpha+\beta} t^{\alpha+\beta}, T(0,t) = 0
\]

(12b)

The dimensionless forms of (12a, b) are [35,41]:

\[
\frac{\partial^{(\alpha+\beta)} T(x,t)}{\partial x^{(\alpha+\beta)}} = \frac{\partial^{(\alpha+\beta)} T(x,t)}{\partial x^{(\alpha+\beta)}} = 0
\]

(13a)

\[
\frac{\partial^{\alpha+\beta} T(0,t)}{\partial x^{\alpha+\beta}} = E^{\alpha+\beta} t^{\alpha+\beta}, T(0,t) = 0
\]

(13b)

Based on eq. (12a), the local fractional model for 1-D fractal heat-conduction in a fractal semi-infinite bar with a source term \( g(x,t) \) is:

\[
K^{2+\alpha+2\beta} \frac{d^{2+\alpha+2\beta} T(x,t)}{dx^{2+\alpha+2\beta}} - \rho^{\alpha+\beta} c^{\alpha+\beta} \frac{d^{2+\alpha+2\beta} T(x,t)}{dx^{2+\alpha+2\beta}} = g(x,t), \quad -\infty < x < \infty, t > 0
\]

(14a)

With

\[
T(x,0) = f(x), -\infty < x < \infty,
\]

(14b)

The dimensionless form of the model (14a, b) is:

\[
\frac{\partial^{2+\alpha+2\beta} T(x,t)}{\partial x^{2+\alpha+2\beta}} = \frac{\partial^{(\alpha+\beta)} T(x,t)}{\partial t^{(\alpha+\beta)}} = 0, \quad -\infty < x < \infty, t > 0
\]

(15a)

\[
T(x,0) = f(x), -\infty < x < \infty,
\]

(15b)

4. Solutions by the Generalized Yang-Fourier transform method:

Let us consider that \( F_{\alpha, \beta} \{ T(x,t) \} = T_0^{F_{\alpha, \beta}}(\omega, t) \) is the Generalized Yang-Fourier transform of \( T(x,t) \), regarded as a non-differentiable function of \( x \). Applying the Yang-Fourier transform to the first term of eq. (15a), we obtain:
\[
F_{\alpha, \beta} \left( \frac{\partial^2 \alpha + \beta}{\partial x^2 \alpha + \beta} T(x,t) \right) = \left( \frac{\partial^2 \alpha + \beta}{\partial t^2 \alpha + \beta} \right) T_{\omega}^{\alpha, \beta} (\omega, t) = \omega^{2(\alpha + \beta)} T_{\omega}^{\alpha, \beta} (\omega, t)
\]  
(16a)

On the other hand, by changing the order of the local fractional differentiation and integration in the second term of eq. (15a), we get:

\[
F_{\alpha, \beta} \left( \frac{\partial \alpha + \beta}{\partial t \alpha + \beta} T(x,t) \right) = \frac{\partial \alpha + \beta}{\partial t \alpha + \beta} T_{\omega}^{\alpha, \beta} (\omega, t)
\]  
(16b)

For the initial value condition, the Yang-Fourier transform provides:

\[
F_{\alpha, \beta} \{ T(x,0) \} = T_{\omega}^{\alpha, \beta} (\omega, 0) = F_{\alpha, \beta} \{ f(x) \} = f_{\omega}^{\alpha, \beta} (\omega)
\]  
(16c)

Thus we get from eqns. (16a,c):

\[
\frac{\partial \alpha + \beta}{\partial t \alpha + \beta} T_{\omega}^{\alpha, \beta} (\omega, t) + \omega^{2(\alpha + \beta)} T_{\omega}^{\alpha, \beta} (\omega, t) = 0, T_{\omega}^{\alpha, \beta} (\omega, 0) = f_{\omega}^{\alpha, \beta} (\omega)
\]  
(17)

This is an initial value problem of a local fractional differential equation with \( t \) as independent variable and \( w \) as a parameter.

\[
T(\omega, t) = \int_{-\infty}^{\infty} E_{\alpha, \beta} \left( \frac{\partial \alpha + \beta}{\partial t \alpha + \beta} \right) e_{\omega}^{\alpha, \beta} (\omega) \left( -\omega^{2(\alpha + \beta)} t^{\alpha + \beta} \right) (d\omega)^{\alpha + \beta}
\]  
(18a)

Consequently, using inversion formula, eqn. (3), we obtain:

\[
T(x,t) = \frac{1}{(2\pi)^{\alpha + \beta}} \int_{-\infty}^{\infty} E_{\alpha, \beta} \left( \frac{\partial \alpha + \beta}{\partial t \alpha + \beta} \right) e_{\omega}^{\alpha, \beta} (\omega) \left( -\omega^{2(\alpha + \beta)} t^{\alpha + \beta} \right) (d\omega)^{\alpha + \beta}
\]  
(18b)

\[
M_{\omega}^{F, \alpha, \beta} (\omega) = \frac{1}{(2\pi)^{\alpha + \beta}} E_{\alpha, \beta} \left( -\omega^{2(\alpha + \beta)} t^{\alpha + \beta} \right)
\]  
(18c)

From [30, 32] we obtained,

\[
F_{\alpha, \beta} \left( E_{\alpha, \beta} \left( -\omega^{2(\alpha + \beta)} \right) \right) = C_{\alpha + \beta, \alpha + \beta} \frac{\omega^{\alpha + \beta}}{\Gamma(1 + \alpha + \beta)} \left( -\omega^{2(\alpha + \beta)} t^{\alpha + \beta} \right)
\]  
(19a)

Let \( C_{\alpha + \beta, \alpha + \beta} / 4^{\alpha + \beta} = t^{\alpha + \beta} \). Then we get:

\[
F_{\alpha, \beta} \left( E_{\alpha, \beta} \left( -\omega^{2(\alpha + \beta)} \right) \right) = \frac{\alpha + \beta}{4^{\alpha + \beta} t^{\alpha + \beta}} \left( \Gamma(1 + \alpha + \beta) \right) \left( -\omega^{2(\alpha + \beta)} t^{\alpha + \beta} \right) = \frac{\alpha + \beta}{4^{\alpha + \beta} t^{\alpha + \beta}} \left( \frac{2\pi}{\alpha + \beta} M_{\omega}^{F, \alpha, \beta} (\omega) \right)
\]  
(19b)

Thus, \( M_{\omega}^{F, \alpha, \beta} (\omega) \) have the inverse:

\[
\frac{1}{(2\pi)^{\alpha + \beta}} \int_{-\infty}^{\infty} E_{\alpha, \beta} \left( \frac{\partial \alpha + \beta}{\partial t \alpha + \beta} \right) M_{\omega}^{F, \alpha, \beta} (\omega) (d\omega)^{\alpha + \beta} = \frac{1}{(2\pi)^{\alpha + \beta} t^{\alpha + \beta}} \left( \frac{2\pi}{\alpha + \beta} \right) E_{\alpha, \beta} \left( -\omega^{2(\alpha + \beta)} t^{\alpha + \beta} \right)
\]  
(19c)

Hence we get:

\[
T(x,t) = \left( M_{\omega} \right) (x) = \frac{1}{(2\pi)^{\alpha + \beta} t^{\alpha + \beta}} \int_{-\infty}^{\infty} E_{\alpha, \beta} \left( \frac{\partial \alpha + \beta}{\partial t \alpha + \beta} \right) f_{\omega}^{\alpha, \beta} (\omega) (d\omega)^{\alpha + \beta}
\]  
(20)

Special case:
If we take \( \beta = 0 \) then the results of generalized Yang Fourier Transforms convert in Yang Fourier Transforms results [41]

Conclusions:
The communication, presented an analytical solution of 1-D heat conduction in fractal semi-infinite bar by the Generalized Yang-Fourier transform of non-differentiable functions.

REFERENCES


