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Double Dirichlet Average of Generalized Multi-Index Mittag-Leffler Function Via Fractional Calculus

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Abstract:- ***The object of the present paper is to establish the results of double Dirichlet average of Generalized Multi-Index Mittag-Leffler Function, using Riemann-Liouville Fractional Integral. The Generalized Multi-Index Mittag-Leffler Function can be measured as a Dirichlet average and connected with fractional calculus. In this paper the solution is to obtained in compact form of double Dirichlet average of Generalized Multi-Index Mittag-Leffler Function as well as conversion into single Dirichlet average of Generalized Multi-Index Mittag-Leffler Function, using fractional integral. The special cases of our results are same as earlier obtained by Saxena, Pogany, Ram and Daiya [23], for single Dirichlet average of Generalized Multi-Index Mittag-Leffler Function.***

Keywords and Phrases: Dirichlet averages, special functions, Generalized Multi-Index Mittag-Leffler Function and Riemann-Liouville Fractional Integral.

Mathematics Subject Classification: 2000: Primary: 33E12, 26A33; Secondary: 33C20, 33C65.

1. Introduction:

Carlson [2-5] has defined Dirichlet averages of functions which represents certain type of integral average with respect to Dirichlet measure. He showed that various important special functions can be derived as Dirichlet averages for the ordinary simple functions likex^t,e^x etc. He has also pointed out [3] that the hidden symmetry of all special functions which provided their various transformations can be obtained by averaging x^n,e^x etc. Thus he established a unique process towards the unification of special functions by averaging a limited number of ordinary functions. Almost all known special functions and their well known properties have been derived by this process.

Gupta and Agarwal [11, 12] found that averaging process is not altogether new but directly connected with the old theory of fractional derivative. Carlson overlooked this connection whereas he has applied fractional derivative in so many cases during his entire work. Deora and Banerji [7] have found the double Dirichlet average of e^x by using fractional derivatives and they have also found the Triple Dirichlet Average of x^t by using fractional derivatives [8].

Sharma and Jain [26] obtained double Dirichlet average of Trigonometry function $\cos x$ using fractional derivative and they have also found the Triple Dirichlet Average of e^x by using fractional calculus.

Recently, Kilbas and Kattuveetti [13] established a correlation among Dirichlet averages of the generalized Mittag-Leffler function with Riemann-Liouville fractional integrals and of the hyper- geometric functions of many variables.

In the present paper the Double Dirichlet average of Generalized Multi-Index Mittag-Leffler Function has been obtained in terms of Riemann-Liouville Fractional integrals. Also the correlation between Double Dirichlet average of Generalized Multi-Index Mittag-Leffler Function with Riemann-Liouville Fractional integrals converted into the correlation between single Dirichlet average of Generalized Multi-Index Mittag-Leffler Function using Riemann-Liouville Fractional integrals.

2. Definitions and Preliminaries:

Some definitions which are necessary in the preparation of this paper.

2.1 Standard Simplex in $R^k, k \geq 1$:

The standard simplex in R^k, $k \geq 1$ by [1, p.62].

$$E = E_k = \{S(u_{1,}u_2, \ldots u_k) : u_1 \geq 0, \ldots u_k \geq 0,\ u_1 + u_2 + \cdots + u_k \leq 1\}$$

2.2 Dirichlet measure:

Let $b \in C^k, k \geq 2$ and let $E = E_{k-1}$ be the standard simplex in R^{k-1}. The complex measure μ_b is defined by E[1].

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$$d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1-1} \dots u_{k-1}^{b_{k-1}-1}(1-u_1-\dots-u_{k-1})b_k^{-1} du_1 \dots du_{k-1}$$

Will be called a Dirichlet measure.
Here

$$B(b) = B(b1, \dots bk) = \frac{\Gamma(b_1)\dots\Gamma(b_k)}{\Gamma(b_1+\dots+b_k)},$$
$$C_> = \{z \in z: z \neq 0, |ph\, z| < {}^{\pi}/_{2}\},$$

Open right half plane and $C_>$k is the k^{th} Cartesian power of $C_>$

2.3 Dirichlet Average[1, p.75]:

Let Ω be the convex set in $C_>$, let $z = (z_1, \dots, z_k) \in \Omega^k$, $k \geq 2$ and let $u.z$ be a convex combination of $z_1, \dots, z_k$. Let f be a measureable function on Ω and let μ_b be a Dirichlet measure on the standard simplex E in R^{k-1}.Define

$$F(b,z) = \int_E f(u.z) d\,\mu_b(u) \tag{2.3}$$

F is the Dirichlet measure of f with variables
$z = (z_1, \dots, z_k)$ and parameters $b = (b_1, \dots b_k)$.
Here

$$u.z = \sum_{i=1}^{k} u_i z_i \text{ and } u_k = 1 - u_1 - \dots - u_{k-1.}$$

If $k = 1$, define $F(b,z) = f(z)$.

The following notation have been used in present work,

$R^k = k^{th}$, Cartesian product of $C_>$,
R = Set of real numbers,
$C_>$ = Open right half plane,
μ_b = Complex measure,
$\Omega^k = k^{th}$, Cartesian product of Ω
Ω = Convex set in $C_>$,
$B(b)$ = Beta function
E = Standard simplex

2.4 Fractional Integral [9, p.181]:

The theory of fractional integral with respect to an arbitrary function has been used by Erdelyi[9]. The general definition for the fractional integral of order α found in the literature on the "Riemann-Liouville integral" is

$$D_z^{\alpha} F(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z F(t)(z-t)^{-\alpha-1} dt \tag{2.4}$$

Where $Re(\alpha) < 0$ and $F(x)$ is the form of $x^p f(x)$, where $f(x)$ is analytic at $x = 0$.

2.5 Average of function $f(x)$ (from [5]):

let μ^b be a Dirichlet measure on the standard simplex E in R^{k-1}; k≥ 2. For every $z \in C^k$

$$S(b,z) = \int_E f(u.z) d\mu_b\,(u) \tag{2.5}$$

If $k = 1, S = (b,z) = f(u.z)$.

2.6 Double averages of functions of one variable (from [2, 3]):

Let z be a $k \times x$ matrix with complex elements z_{ij}. Let $u = (u_1, \dots u_k) and\ v = (v_1, \dots v_k)$ be an ordered k-tuple and x-tuple of real non-negative weights $\sum u_i = 1\ and\ \sum v_j = 1$, respectively.
We define

$$u.z.v = \sum_{i=1}^{k}\sum_{j=1}^{x} u_i z_{ij} v_j \tag{2.6}$$

If z_{ij} is regarded as a point of the complex plane, all these convex combinations are points in the convex hull of $(z_{11}, \ldots z_{kx})$, denote by $H(z)$.

Let $b = (b_1, \ldots b_k) be$ an ordered $k-$tuple of complex numbers with positive real part (Re(b) > 0) and similarly for β=$(\beta_1, \ldots \beta_x)$. Then we define $d\mu_b(u) and\ d\mu_b(v)$.

Let f be the holomorphic on a domain D in the complex plane, If $Re(b) > 0, Re(\beta) > 0 and H(z) \subset D$, we define

$$F(b, z, \beta) = \iint f(u, z, v)\, d\mu_b(u) d\mu_b(v) \tag{2.7}$$

Generalized Multi-Index Mittag-Leffler Function:

The generalized multi-index Mittag-Leffler function defined and studied by Saxena and Nishimoto [21, 22].

$$E_{\sigma,k}[(\alpha_i, \delta_i)_{1,m}; z] = \sum_{k=0}^{\infty} \frac{(\sigma)_{kn} z^k}{\prod_{j=1}^{m} \Gamma(\alpha_i + \delta_i n)\, n!} \tag{2.8}$$

3. Main Results and Proof:

Theorem: Following equivalence relation for Double Dirichlet Average is established for $(k = x = 2)$ of the generalized multi-index Mittag-Leffler function.

$$S[(\alpha_i, \delta_i)_{1,m};\ \mu, \mu'; z; \rho, \rho'] = \frac{\Gamma(\rho + \rho')}{\Gamma\rho} \frac{(\mu)_k}{(\mu + \mu')_k} (x - y)^{1-\rho-\rho'}$$

$$D_{x-y}^{-\rho'} E_{\sigma,k}[(\alpha_i, \delta_i)_{1,m}; (x)]\ (x - y)^{\rho-1} \tag{3.1}$$

Proof:

Let us consider the double average for $(k = x = 2$ of of the generalized multi-index Mittag-Leffler function.

$$S[(\alpha_i, \delta_i)_{1,m};\ \mu, \mu'; z; \rho, \rho'] = \sum_{k=0}^{\infty} \frac{(\sigma)_{kn}}{\prod_{j=1}^{m} \Gamma(\alpha_i + \delta_i n)\, n!} \int_0^1 \int_0^1 [u.z.v]^k\, dm_{\mu,\mu'}(u) dm_{\rho,\rho'}(v)$$

$$= \sum_{k=0}^{\infty} \frac{(\sigma)_{kn}}{\prod_{j=1}^{m} \Gamma(\alpha_i + \delta_i n)\, n!} \int_0^1 \int_0^1 [u.z.v]^k dm_{\mu,\mu'}(u) dm_{\rho,\rho'}(v)$$

$Re(\mu) = 0, Re(\mu') = 0, Re(\rho) > 0, Re(\rho') > 0$ and

$$u.z.v = \sum_{i=1}^{2}\sum_{i=1}^{2} (u_i z_{ij} v_j) = \sum_{i=1}^{2} [u_i (z_{i1} v_1 + z_{i2} v_2)]$$

$$= [u_1 z_{11} v_1 + u_1 z_{12} v_2 + u_2 z_{21} v_1 + u_2 z_{22} v_2]$$

let $z_{11} = a, z_{12} = b, z_{21} = c, z_{22} = d$ and $\begin{cases} u_1 = u, & u_2 = 1 - u \\ v_1 = v, & v_2 = 1 - v \end{cases}$

thus $z = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$v.z.v = uva + ub(1 - v) + (1 - u)cv + (1 - u)d(1 - v)$$

$$= uv(a - b - c + d) + u(b - d) + v(c - d) + d$$

$$dm_{\mu,\mu'}(u) = \frac{\Gamma(\mu + \mu')}{\Gamma\mu\, \Gamma\mu'} u^{\mu-1}(1 - u)^{\mu'-1} du$$

$$dm_{\rho,\rho'}(v) = \frac{\Gamma(\rho + \rho')}{\Gamma\rho\Gamma\rho'} v^{\rho-1}(1 - v)^{\rho'-1} dv$$

Putting these values in (3.1), we have,

$$S[(\alpha_i,\delta_i)_{1,m};\ \mu,\mu';z;\rho,\rho'] = \frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\sum_{k=0}^{\infty}\frac{(\sigma)_{kn}}{\prod_{j=1}^{m}\Gamma(\alpha_i+\delta_i n)\,n!} \times \int_0^1\int_0^1 [uv(a-b-c+d)+u(b-d)+v(c-d)+d]^k \, u^{\mu-1}(1-u)^{\mu'-1}v^{\rho-1}(1-v)^{\rho'-1}dudv$$

In order to obtained the fractional derivative equivalent to the above integral,
Case –I: we assume $a = x, b = y$ and $c = d = 0$ then

$$S[(\alpha_i,\delta_i)_{1,m};\ \mu,\mu';z;\rho,\rho'] = \frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\sum_{k=0}^{\infty}\frac{(\sigma)_{kn}}{\prod_{j=1}^{m}\Gamma(\alpha_i+\delta_i n)\,n!} \times \int_0^1\int_0^1 [uv(x-y)+uy]^k \, u^{\mu-1}(1-u)^{\mu'-1}v^{\rho-1}(1-v)^{\rho'-1}dudv$$

We use the pochammer symbols and beta function, we have

$$S[(\alpha_i,\delta_i)_{1,m};\ \mu,\mu';z;\rho,\rho'] = \frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\frac{(\mu)_k}{(\mu+\mu')_k}\sum_{k=0}^{\infty}\frac{(\sigma)_{kn}}{\prod_{j=1}^{m}\Gamma(\alpha_i+\delta_i n)\,n!} \times \int_0^1 [v(x-y)+y]^k \, v^{\rho-1}(1-v)^{\rho'-1}dv$$

Putting $v(x-y) = t$, we obtain

$$S[(\alpha_i,\delta_i)_{1,m};\ \mu,\mu';z;\rho,\rho'] = \frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\frac{(\mu)_k}{(\mu+\mu')_k}\sum_{k=0}^{\infty}\frac{(\sigma)_{kn}}{\prod_{j=1}^{m}\Gamma(\alpha_i+\delta_i n)\,n!} \times \int_0^{x-y} [y+t]^k \left(\frac{t}{x-y}\right)^{\rho-1}\left(1-\frac{t}{x-y}\right)^{\rho'-1}\frac{dt}{(x-y)}$$

$$S[(\alpha_i,\delta_i)_{1,m};\ \mu,\mu';z;\rho,\rho'] = \frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\frac{(\mu)_k}{(\mu+\mu')_k}\sum_{k=0}^{\infty}\frac{(\sigma)_{kn}}{\prod_{j=1}^{m}\Gamma(\alpha_i+\delta_i n)\,n!} \times (x-y)^{1-\rho-\rho'}\int_0^{x-y} [y+t]^k (t)^{\rho-1}(x-y-t)^{\rho'-1}dt$$

$$S[(\alpha_i,\delta_i)_{1,m};\ \mu,\mu';z;\rho,\rho'] = \frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\frac{(\mu)_k}{(\mu+\mu')_k}(x-y)^{1-\rho-\rho'} \int_0^{x-y} E_{\sigma,k}[(\alpha_i,\delta_i)_{1,m};(y+t)]\,(t)^{\rho-1}(x-y-t)^{\rho'-1}dt \qquad (3.2)$$

Using definition of fractional derivative (2.4), we get

$$S[(\alpha_i,\delta_i)_{1,m};\ \mu,\mu';z;\rho,\rho'] = \frac{\Gamma(\rho+\rho')}{\Gamma\rho}\frac{(\mu)_k}{(\mu+\mu')_k}(x-y)^{1-\rho-\rho'} D_{x-y}^{-\rho'}E_{\sigma,k}[(\alpha_i,\delta_i)_{1,m};(x)]\,(x-y)^{\rho-1} \qquad (3.3)$$

Case II: If we assume $a = c = x; b = d = y$ then the double Dirichlet average of the generalized multi-index Mittag-Leffler function converted into single Dirichlet average of that functions, we have

$$S[(\alpha_i,\delta_i)_{1,m};\ \mu,\mu';z;\rho,\rho'] = \frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\sum_{k=0}^{\infty}\frac{(\sigma)_{kn}}{\prod_{j=1}^{m}\Gamma(\alpha_i+\delta_i n)\,n!}$$

$$\times \int_0^1 \int_0^1 [v(x-y)+y]^k u^{\mu-1}(1-u)^{\mu'-1} v^{\rho-1}(1-v)^{\rho'-1} du dv$$

$$S[(\alpha_i,\delta_i)_{1,m};\ \mu,\mu';z;\rho,\rho'] = \frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'} \sum_{k=0}^{\infty} \frac{(\sigma)_{kn}}{\prod_{j=1}^{m}\Gamma(\alpha_i+\delta_i n)\, n!}$$

$$\times \int_0^1 f[v(x-y)+y]^k v^{\rho-1}(1-v)^{\rho'-1} dv$$

Putting $v(x-y)=t$, we obtain

$$S[(\alpha_i,\delta_i)_{1,m};\ \mu,\mu';z;\rho,\rho'] = \frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'} \sum_{k=0}^{\infty} \frac{(\sigma)_{kn}}{\prod_{j=1}^{m}\Gamma(\alpha_i+\delta_i n)\, n!}$$

$$\times \int_0^{x-y} [y+t]^k \left(\frac{t}{x-y}\right)^{\rho-1} \left(1-\frac{t}{x-y}\right)^{\rho'-1} \frac{dt}{(x-y)}$$

$$S[(\alpha_i,\delta_i)_{1,m};\ \mu,\mu';z;\rho,\rho'] = \frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'} \sum_{k=0}^{\infty} \frac{(\sigma)_{kn}}{\prod_{j=1}^{m}\Gamma(\alpha_i+\delta_i n)\, n!} (x-y)^{1-\rho-\rho'}$$

$$\int_0^{x-y} [y+t]^k (t)^{\rho-1}(x-y-t)^{\rho'-1} dt$$

$$S[(\alpha_i,\delta_i)_{1,m};\ \mu,\mu';z;\rho,\rho'] = \frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'} (x-y)^{1-\rho-\rho'}$$

$$\int_0^{x-y} E_{\sigma,k}[(\alpha_i,\delta_i)_{1,m};(y+t)]\,(t)^{\rho-1}(x-y-t)^{\rho'-1} dt \qquad (3.4)$$

Using definition of fractional integral (2.4), we get

$$S[(\alpha_i,\delta_i)_{1,m};\ \mu,\mu';z;\rho,\rho'] = \frac{\Gamma(\rho+\rho')}{\Gamma\rho} (x-y)^{1-\rho-\rho'} D_{x-y}^{-\rho'} E_{\sigma,k}[(\alpha_i,\delta_i)_{1,m};(x)]\,(x-y)^{\rho-1} \qquad (3.5)$$

This is complete proof of (3.1). Thus the above result is same as earlier derived by Saxena, Pogany, Ram and Daiya [23]

Special cases:

(i). If $k=1, \sigma=1$, then the result as follows:

$$S[(\alpha_i,\delta_i)_{1,m};\ \mu,\mu';z;\rho,\rho'] = \frac{\Gamma(\rho+\rho')}{\Gamma\rho} \frac{(\mu)_1}{(\mu+\mu')_1} (x-y)^{1-\rho-\rho'}$$

$$D_{x-y}^{-\rho'} E_{1,1}[(\alpha_i,\delta_i)_{1,m};(x)]\,(x-y)^{\rho-1} \qquad (3.6)$$

If $k=1, \sigma=1$, then the result is converted into single Direchlet average of the generalized multi-index Mittag-Leffler function, which is an special case of Sexena, et.al.[23]

$$S[(\alpha_i,\delta_i)_{1,m};\ \mu,\mu';z;\rho,\rho'] = \frac{\Gamma(\rho+\rho')}{\Gamma\rho} (x-y)^{1-\rho-\rho'} D_{x-y}^{-\rho'} E_{1,1}[(\alpha_i,\delta_i)_{1,m};(x)]\,(x-y)^{\rho-1} \qquad (3.7)$$

(ii). If we assume $k=1=m, \alpha_1=\alpha$, and $\delta_1=\delta$, then the result as under:

$$S[(\alpha,\delta)_{1,1};\ \mu,\mu';z;\rho,\rho'] = \frac{\Gamma(\rho+\rho')}{\Gamma\rho} \frac{(\mu)_1}{(\mu+\mu')_1} (x-y)^{1-\rho-\rho'}$$

$$D_{x-y}^{-\rho'} E_{\sigma,1}[(\alpha,\delta)_{1,1};(x)]\,(x-y)^{\rho-1} \qquad (3.8)$$

If we assume $k=1=m, \alpha_1=\alpha$, and $\delta_1=\delta$, then the result reduces to a single Direchlet average of the generalized multi-index Mittag-Leffler function.

$$S[(\alpha,\delta)_{1,1};\ \mu,\mu';z;\rho,\rho'] = \frac{\Gamma(\rho+\rho')}{\Gamma\rho}(x-y)^{1-\rho-\rho'}D_{x-y}^{-\rho'}E_{\sigma,1}[(\alpha,\delta)_{1,1};(x)]\,(x-y)^{\rho-1} \quad (3.9)$$

6. Conclusion:

Thus, we can say that every analytic function can be measured as double Dirichlet average, using fractional integral. Also, the relation between double Dirichlet average of any analytic functions and fractional integral can be converted into single Dirichlet average of those functions, using fractional integrals of the functions. This fact we have shown by solving the generalized multi-index Mittag-Leffler function. Thus the obtained result for double Dirichlet average of generalized multi-index Mittag-Leffler function is generalized results of earlier derived by Saxena, Pogany, Ram and Daiya [23] .

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