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Generalized Yang-Fourier Transforms to Heat-Conduction in a Semi-Infinite Fractal Bar

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Abstract:- The aim of present paper to solve 1-D fractal heat-conduction problem in a fractal semi-infinite bar has been developed by local fractional calculus employing the analytical Generalized Yang-Fourier transforms method.

Keywords: fractal bar, heat-conduction equation, Generalized Yang-Fourier transforms, Yang-Fourier transforms, local fractional calculus.

1. Introduction:

Generalized Yang-Fourier transforms which is obtained by authors by generalization of Yang-Fourier transforms is a technique of fractional calculus for solving mathematical, physical and engineering problems. The fractional calculus is continuously growing in last five decades [1-7]. Most of the fractional ordinary differential equations have exact analytic solutions, while others required either analytical approximations or numerical techniques to be applied, among them: fractional Fourier and Laplace transforms [8,41], heat-balance integral method [9-11], variation iteration method (VIM) [12-14], decomposition method [15,41], homotopy perturbation method [16,41] etc.

The problems in fractal media can be successfully solved by local fractional calculus theory with problems for non-differential functions [25-32]. Local fractional differential equations have been applied to model complex systems of fractal physical phenomena [30-41] local fractional Fourier series method [38], Yang-Fourier transform [39, 40,41]

2. Generalized Yang-Fourier transform and its properties:

Let us Consider $f(x)$ is local fractional continuous in $(-\infty,\infty)$ we denote as $f(x) \in C_{\alpha,\beta}(-\infty,\infty)$ [32, 33, 35].

Let $f(x) \in C_{\alpha,\beta}(-\infty,\infty)$ The Generalized Yang-Fourier transform developed by authors is written in the form [30, 31, 39, 40, 41]:

$$F_{\alpha,\beta}\{f(x)\} = f_{\omega}^{F,\alpha,\beta}(\omega) = \frac{1}{\Gamma(1+\alpha+\beta)} \int_{-\infty}^{\infty} E_{\alpha,\beta}\left(-i^{\alpha+\beta}\omega^{\alpha+\beta}x^{\alpha+\beta}\right) f(x)(dx)^{\alpha+\beta} \qquad (1)$$

When we put β equal to zero, it converts in to the Yang-Fourier transform [41].

Then, the local fractional integration is given by [30-32, 35-37, 41]:

$$\frac{1}{\Gamma(1+\alpha+\beta)} \int_{a}^{b} f(t)\,(dx)^{\alpha+\beta} = \frac{1}{\Gamma(1+\alpha+\beta)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j)\,(\Delta t_j)^{\alpha+\beta} \qquad (2)$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = max\{\Delta t_1, \Delta t_2, \Delta t_j, \ldots\}$ and $\{t_j, t_{j+1}\}$, $j = 0, \ldots, N-1$, $t_0 = a$, $t_N = b$, is a partition of the interval $[a, b]$.

If $F_{\alpha,\beta}\{f(x)\} = f_{\omega}^{F,\alpha,\beta}(\omega)$, then its inversion formula takes the form [30, 31, 39, 40,41]

$$f(x) = F_{\alpha,\beta}^{-1}\left[f_{\omega}^{F,\alpha,\beta}(\omega)\right] = \frac{1}{(2\pi)^{\alpha+\beta}} \int_{-\infty}^{\infty} E_{\alpha,\beta}\left(-i^{\alpha+\beta}\omega^{\alpha+\beta}x^{\alpha+\beta}\right) f_{\omega}^{F,\alpha,\beta}(\omega)(d\omega)^{\alpha+\beta} \qquad (3)$$

When we put β equal to zero, it converts in to the Yang Inverse Fourier transform [41].

Some properties are shown as it follows [30, 31]:

Let $F_{\alpha,\beta}\{f(x)\} = f_{\omega}^{F,\alpha,\beta}(\omega)$, and $F_{\alpha,\beta}\{g(x)\} = f_{\omega}^{F,\alpha,\beta}(\omega)$, and let be two constants. Then we have:

$$F_{\alpha,\beta}\{cf(x) + dg(x)\} = cF_{\alpha,\beta}\{f(x)\} + dF_{\alpha,\beta}\{g(x)\} \tag{4}$$

If $\lim_{|x|\to\infty} f(x) = 0$, then we have:

$$F_{\alpha,\beta}\{f^{\alpha,\beta}(x)\} = i^{\alpha+\beta}\omega^{\alpha+\beta}F_{\alpha,\beta}\{f(x)\} \tag{5}$$

In eq. (5) the local fractional derivative is defined as:

$$f^{\alpha,\beta}(x_0) = \left.\frac{d^{\alpha+\beta}f(x)}{dx^{\alpha+\beta}}\right|_{x=x_0} = \lim_{x\to x_0}\frac{\Delta^{\alpha+\beta}[f(x)-f(x_0)]}{(x-x_0)^{\alpha+\beta}} \tag{6}$$

Where $\Delta^{\alpha+\beta}[f(x)-f(x_0)] \cong \Gamma(1+\alpha+\beta)\Delta[f(x)-f(x_0)]$

As a direct result, repeating this process, when:

$$f(0) = f^{\alpha,\beta}(0) = \cdots = f^{(k-1)\alpha,(k-1)\beta}(0) = 0 \tag{7}$$

$$F_{\alpha,\beta}\{f^{k\alpha,k\beta}(x)\} = i^{\alpha+\beta}\omega^{\alpha+\beta}F_{\alpha,\beta}\{f(x)\} \tag{8}$$

3. Heat conduction in a fractal semi-infinite bar:

If a fractal body is subjected to a boundary perturbation, then the heat diffuses in depth modeled by a constitutive relation where the rate of fractal heat flux $\overline{q}(x,y,z,t)$ is proportional to the local fractional gradient of the temperature [32,41], namely:

$$\overline{q}(x,y,z,t) = -K^{2\alpha+2\beta}\nabla^{\alpha+\beta}T(x,y,z,t) \tag{9}$$

Here the pre-factor $K^{2a\ 2\beta}$ is the thermal conductivity of the fractal material. Therefore, the fractal heat conduction equation without heat generation was suggested in [32] as:

$$K^{2\alpha+2\beta}\frac{d^{2(\alpha+\beta)}T(x,y,z,t)}{dx^{2(\alpha+\beta)}} - \rho_{\alpha+\beta}c_{\alpha+\beta}\frac{d^{2(\alpha+\beta)}T(x,y,z,t)}{dx^{2(\alpha+\beta)}} = 0 \tag{10}$$

Where $\rho_{\alpha+\beta}$ and $c_{\alpha+\beta}$ are the density and the specific heat of material, respectively.

The fractal heat-conduction equation with a volumetric heat generation $g(x,y,z,t)$ can be described as [32,41]:

$$K^{2\alpha+2\beta}\nabla^{2\alpha+2\beta}T(x,y,z,t) + g(x,y,z,t)\rho_{\alpha+\beta}c_{\alpha+\beta}\frac{\partial^{(\alpha+\beta)}T(x,y,z,t)}{\partial t^{(\alpha+\beta)}} \tag{11}$$

The 1-D fractal heat-conduction equation [32,41] reads as:

$$K^{2\alpha+2\beta}\frac{\partial^{2(\alpha+\beta)}T(x,t)}{\partial x^{2(\alpha+\beta)}} - \rho_{\alpha+\beta}c_{\alpha+\beta}\frac{\partial^{(\alpha+\beta)}T(x,t)}{\partial t^{(\alpha+\beta)}} = 0, \quad 0 < x < \infty, t > 0 \tag{12a}$$

with initial and boundary conditions are:

$$\frac{\partial^{(\alpha+\beta)}T(0,t)}{\partial t^{(\alpha+\beta)}} = E_{\alpha+\beta}t^{\alpha+\beta}, T(0,t) = 0 \tag{12b}$$

The dimensionless forms of (12a, b) are [35, 41]:

$$\frac{\partial^{2(\alpha+\beta)}T(x,t)}{\partial x^{2(\alpha+\beta)}} = \frac{\partial^{(\alpha+\beta)}T(x,t)}{\partial x^{(\alpha+\beta)}} = 0 \tag{13a}$$

$$\frac{\partial^{(\alpha+\beta)}T(0,t)}{\partial x^{(\alpha+\beta)}} = E_{\alpha+\beta}t^{\alpha+\beta}, T(0,t) = 0 \tag{13b}$$

Based on eq. (12a), the local fractional model for 1-D fractal heat-conduction in a fractal semi-infinite bar with a source term $g(x,t)$ is:

$$K^{2\alpha+2\beta}\frac{\partial^{2(\alpha+\beta)}T(x,t)}{\partial x^{2(\alpha+\beta)}} - \rho_{\alpha+\beta}c_{\alpha+\beta}\frac{\partial^{(\alpha+\beta)}T(x,t)}{\partial t^{(\alpha+\beta)}} = g(x,t), \quad -\infty < x < \infty, t > 0 \tag{14a}$$

With

$$T(x,0) = f(x), -\infty < x < \infty, \tag{14b}$$

The dimensionless form of the model (14a, b) is:

$$\frac{\partial^{2(\alpha+\beta)}T(x,t)}{\partial x^{2(\alpha+\beta)}} = \frac{\partial^{(\alpha+\beta)}T(x,t)}{\partial t^{(\alpha+\beta)}} = 0, \quad -\infty < x < \infty, t > 0 \tag{15a}$$

$$T(x,0) = f(x), -\infty < x < \infty, \tag{15b}$$

4. Solutions by the Generalized Yang-Fourier transform method:

Let us consider that $F_{\alpha,\beta}\{T(x,t)\} = T_{\omega}^{F,\alpha,\beta}(\omega,t)$ is the Generalized Yang-Fourier transform of $T(x, t)$, regarded as a non-differentiable function of x. Applying the Yang-Fourier transform to the first term of eq. (15a), we obtain:

$$F_{\alpha,\beta}\left\{\frac{\partial^{2(\alpha+\beta)}T(x,t)}{\partial x^{2(\alpha+\beta)}}\right\} = \left(i^{2(\alpha+\beta)}\omega^{2(\alpha+\beta)}\right)T_{\omega}^{F,\alpha,\beta}(\omega,t) = \omega^{2(\alpha+\beta)}T_{\omega}^{F,\alpha,\beta}(\omega,t) \quad (16a)$$

On the other hand, by changing the order of the local fractional differentiation and integration in the second term of eq.(15a), we get:

$$F_{\alpha,\beta}\left\{\frac{\partial^{2(\alpha+\beta)}}{\partial t^{2(\alpha+\beta)}}T(x,t)\right\} = \frac{\partial^{(\alpha+\beta)}}{\partial t^{(\alpha+\beta)}}T_{\omega}^{F,\alpha,\beta}(\omega,t) \quad (16b)$$

For the initial value condition, the Yang-Fourier transform provides:

$$F_{\alpha,\beta}\{T(x,0)\} = T_{\omega}^{F,\alpha,\beta}(\omega,0) = F_{\alpha,\beta}\{f(x)\} = f_{\omega}^{F,\alpha,\beta}(\omega) \quad (16c)$$

Thus we get from eqn.(16a,b,c):

$$\frac{\partial^{(\alpha+\beta)}}{\partial t^{(\alpha+\beta)}}T_{\omega}^{F,\alpha,\beta}(\omega,t) + \omega^{2(\alpha+\beta)}T_{\omega}^{F,\alpha,\beta}(\omega,t) = 0, T_{\omega}^{F,\alpha,\beta}(\omega,0) = f_{\omega}^{F,\alpha,\beta}(\omega) \quad (17)$$

This is an initial value problem of a local fractional differential equation with t as independent variable and w as a parameter.

$$T(\omega,t) = f_{\omega}^{F,\alpha,\beta}(\omega)E_{\alpha,\beta}\left(-\omega^{2(\alpha+\beta)}t^{\alpha+\beta}\right) \quad (18a)$$

Consequently, using inversion formula, eqn. (3), we obtain:

$$T(x,t) = \frac{1}{(2\pi)^{\alpha+\beta}}\int_{-\infty}^{\infty}E_{\alpha,\beta}\left(i^{\alpha+\beta}\omega^{\alpha+\beta}x^{\alpha+\beta}\right)f_{\omega}^{F,\alpha,\beta}(\omega)E_{\alpha,\beta}\left(-\omega^{2(\alpha+\beta)}t^{\alpha+\beta}\right)(d\omega)^{\alpha+\beta} \quad (18b)$$

$$M_{\omega}^{F,\alpha,\beta}(\omega) = \frac{1}{(2\pi)^{\alpha+\beta}}E_{\alpha,\beta}\left(-\omega^{2(\alpha+\beta)}t^{\alpha+\beta}\right) \quad (18c)$$

From [30, 32] we obtained,

$$F_{\alpha+\beta}\left\{E_{\alpha+\beta}\left(-\frac{\omega^{2(\alpha+\beta)}}{C^{2(\alpha+\beta)}}\right)\right\} = \frac{C^{(\alpha+\beta)}\pi^{\frac{\alpha+\beta}{2}}}{\Gamma(1+\alpha+\beta)}E_{\alpha,\beta}\left(-\frac{C^{2(\alpha+\beta)}\omega^{2(\alpha+\beta)}}{4^{(\alpha+\beta)}}\right) \quad (19a)$$

Let $C^{2(\alpha+\beta)}/4^{\alpha+\beta} = t^{\alpha+\beta}$. Then we get:

$$F_{\alpha+\beta}\left\{E_{\alpha+\beta}\left(-\frac{\omega^{2(\alpha+\beta)}}{4^{\alpha+\beta}\,t^{\alpha+\beta}}\right)\right\} = \frac{4^{\alpha+\beta}t^{\frac{\alpha+\beta}{2}}\pi^{\frac{\alpha+\beta}{2}}}{\Gamma(1+\alpha+\beta)}E_{\alpha,\beta}\left(-\omega^{2(\alpha+\beta)}t^{\alpha+\beta}\right) = \frac{4^{\alpha+\beta}t^{\frac{\alpha+\beta}{2}}\pi^{\frac{\alpha+\beta}{2}}}{\Gamma(1+\alpha+\beta)}(2\pi)^{\alpha+\beta}M_{\omega}^{F,\alpha,\beta}(\omega)$$

$$(19b)$$

Thus, $M_{\omega}^{F,\alpha,\beta}(\omega)$ have the inverse:

$$\frac{1}{(2\pi)^{\alpha+\beta}}\int_{-\infty}^{\infty}E_{\alpha,\beta}\left(i^{\alpha+\beta}\omega^{\alpha+\beta}x^{\alpha+\beta}\right)M_{\omega}^{F,\alpha,\beta}(\omega)(d\omega)^{\alpha+\beta} = \frac{\Gamma(1+\alpha+\beta)}{4^{\alpha+\beta}t^{\frac{\alpha+\beta}{2}}\pi^{\frac{\alpha+\beta}{2}}}\frac{1}{(2\pi)^{\alpha+\beta}}E_{\alpha+\beta}\left(-\frac{\omega^{2(\alpha+\beta)}}{4^{\alpha+\beta}\,t^{\alpha+\beta}}\right) \quad (19c)$$

Hence, we get:

$$T(x,t) = (Mf)(x) = \frac{\Gamma(1+\alpha+\beta)}{4^{\alpha+\beta}t^{\frac{\alpha+\beta}{2}}\pi^{\frac{\alpha+\beta}{2}}}\int_{-\infty}^{\infty}f(\xi)E_{\alpha,\beta}\left(-\frac{(x-\xi)^{2(\alpha+\beta)}}{4^{\alpha+\beta}t^{\alpha+\beta}}\right)(d\xi)^{\alpha+\beta} \quad (20)$$

Special case:

If we take $\beta = 0$ then the results of generalized Yang Fourier Transforms convert in Yang Fourier Transforms results [41]

Conclusions:

The communication, presented an analytical solution of 1-D heat conduction in fractal semi-infinite bar by the Generalized Yang-Fourier transform of non-differentiable functions.

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